CIVE 3203

Displacements in Beams & Frames using Virtual Work

N. Holtz
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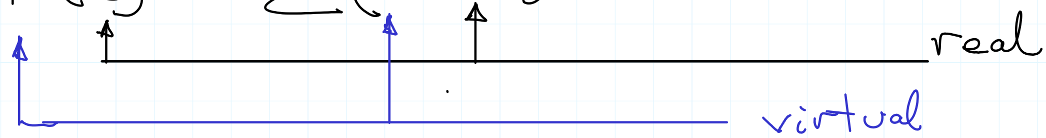
Revision History:
- none

Deflections in Beams & Frames
Method of Virtual Work (Virtual Force)

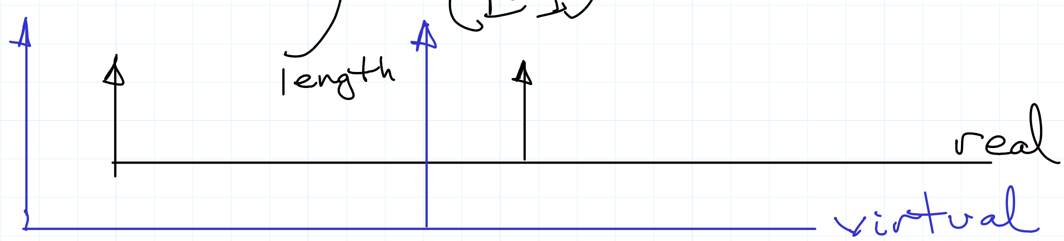
Basic relationship (from before):

$$1\,(\Delta) = \sum (u\;\delta L)$$



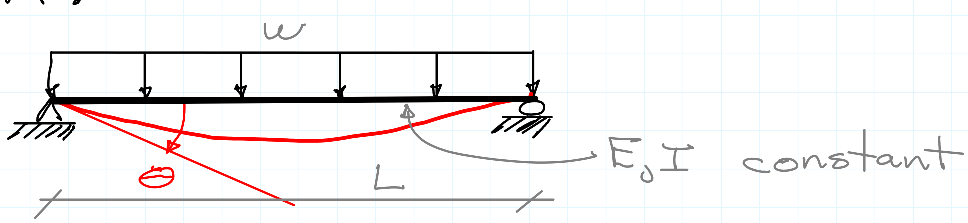
For beams & frames, where the displacements & distortions are due to flexure, the appropriate version of this is:

$$1\,(\Delta) = \int_{\text{length}} m\left(\frac{M}{EI}\right) dx$$

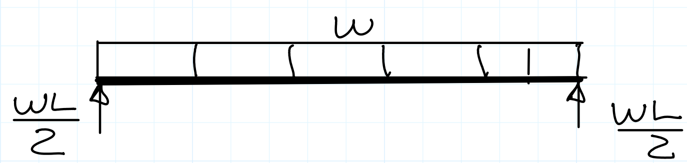


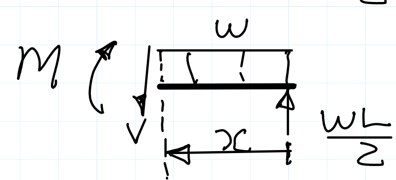
Example

Compute the rotation of the tangent at the support:

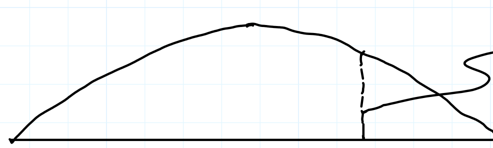


a) construct the real $\frac{M}{EI}$ 'diagram' - entire structure.





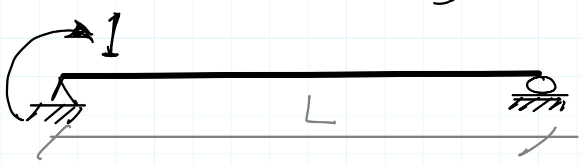
$\frac{M}{EI}$



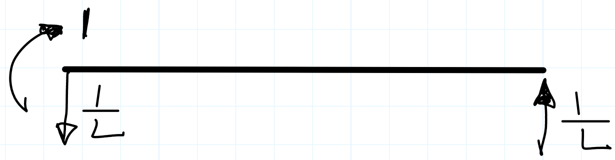
$$\frac{1}{EI}\left(\frac{wL}{2}x - \frac{wx^2}{2}\right)$$

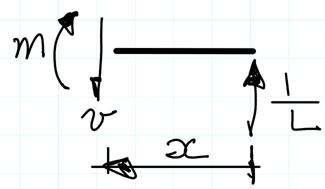
for $0 \leq x \leq L$

b) apply a virtual force corresponding to the desired displacement (in this case, a rotation). Therefore, apply a virtual couple.

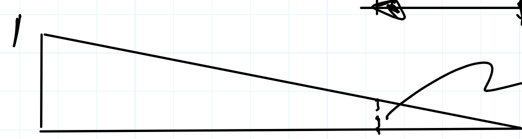


c) determine virtual moments





m



$m = \frac{1}{L}x$

for $0 \le x \le L$

d) compute

$$1 \cdot \theta = \int m \frac{M}{EI}\, dx$$

$$= \frac{1}{EI}\int_0^L \frac{1}{L}x\left(\frac{wL}{2}x - \frac{wx^2}{2}\right) dx$$

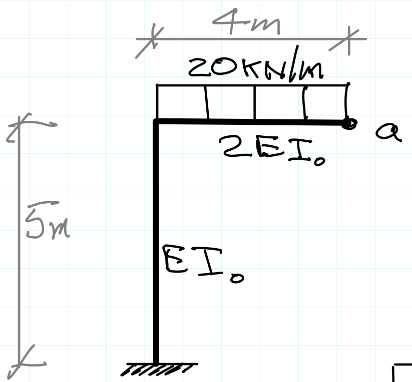
$$= \frac{1}{EI}\int_0^L \left(\frac{wx^2}{2} - \frac{wx^3}{2L}\right) dx$$

$$= \frac{1}{EI}\left[\frac{wx^3}{6} - \frac{wx^4}{8L}\right]_0^L$$

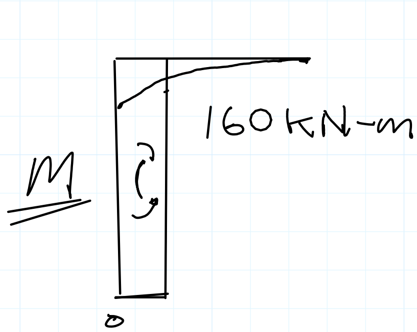
$$= \frac{1}{EI}\left[\frac{wL^3}{6} - \frac{wL^3}{8}\right]$$

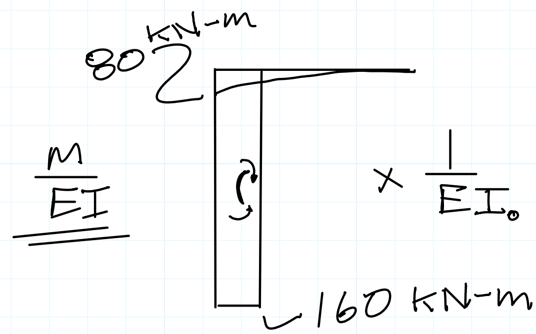
$$\boxed{\theta = \frac{wL^3}{24EI}}$$

Examples using the integral tables.

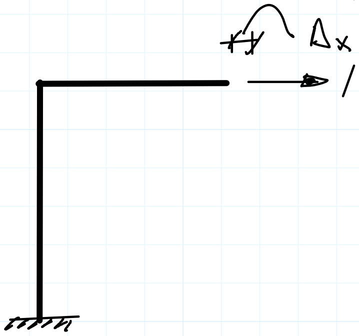


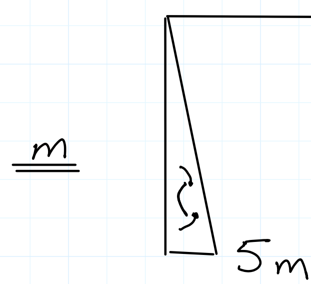
Find horizontal displacement, vertical displacement, and rotation of pt. a.



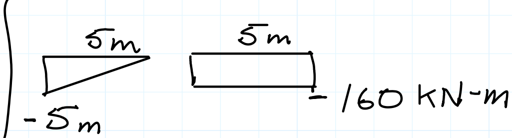


Horizontal Displacement:





$1 \times \Delta_x = \frac{1}{EI_0} \int$

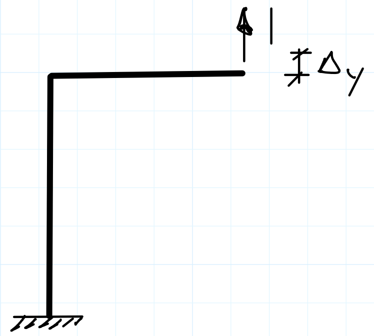


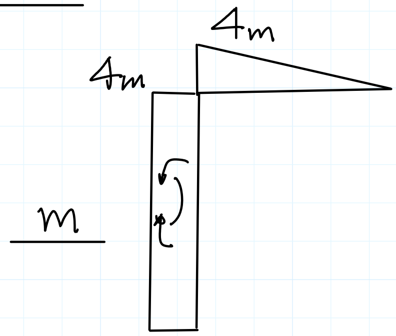
(row 2, col 1 in table)

$= \frac{1}{EI_0} \frac{5}{2} \times -5 \times -160 \text{ kN-m}^3$

$$\Delta_x = \frac{2000 \text{ kN-m}^3}{EI_0}$$

Vertical Displacement





$$1(\Delta_y) = \frac{1}{EI_o}\left[\int_{4m}^{5} \left(\text{rectangle } 5\right)\left(\text{rectangle } -160\ kN\text{-}m\right) + \int_{4}^{4} \left(\text{triangle } 4\right)\left(\text{parabola } -80\right)\right]$$

$$= \frac{1}{EI_o}\left[5 \times 4 \times -160\ kN\text{-}m^3 + \frac{4}{4} \times 4 \times -80\ kN\text{-}m^3\right]$$

(row1, col 1)

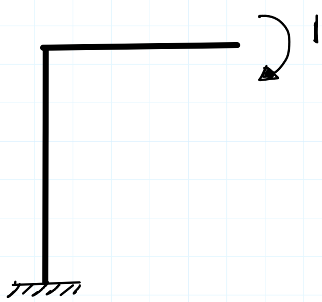
(row3, col 6)

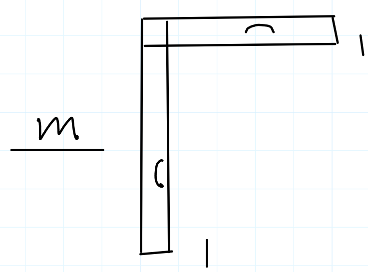
Note col contributes 10x as much as beam

$$\boxed{\Delta_y = \frac{-3520\ kN\text{-}m^3}{EI_o}}$$

$(\therefore \downarrow)$

Rotation



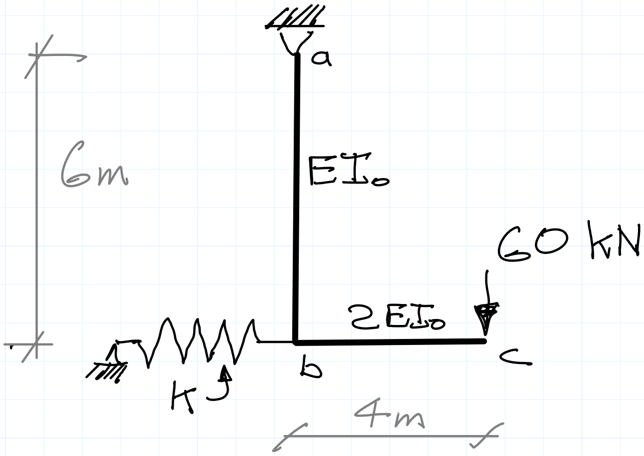


$$1 \times \theta = \frac{1}{EI_o}\left\{\int_{-1}^{5m} \left(\text{rectangle}\right)\left(\text{rectangle } -160\ kN\text{-}m\right) + \int_{-1}^{4} \left(\text{rectangle}\right)\left(\text{parabola } -80\right)\right\}$$

$$= \frac{1}{EI_o}\left\{5 \times -1 \times -160\ kN\text{-}m^2 + \frac{4}{3} \times -1 \times -80\ kN\text{-}m^2\right\}$$

$$\boxed{\theta = \frac{2720\ kN\text{-}m^2}{3\ EI_o}}$$

Combining axial & flexural effects



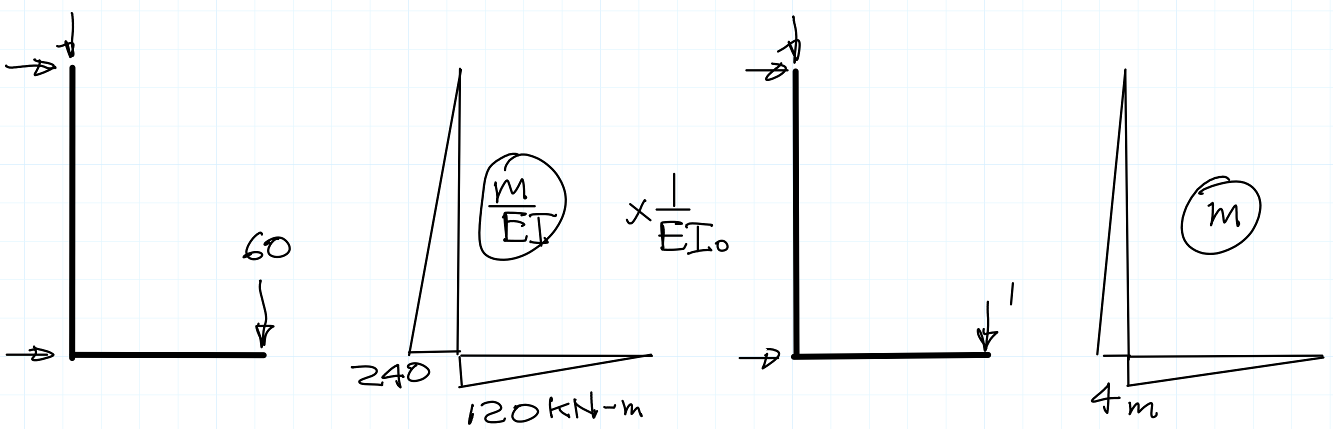
$E = 200\ 000$ MPa

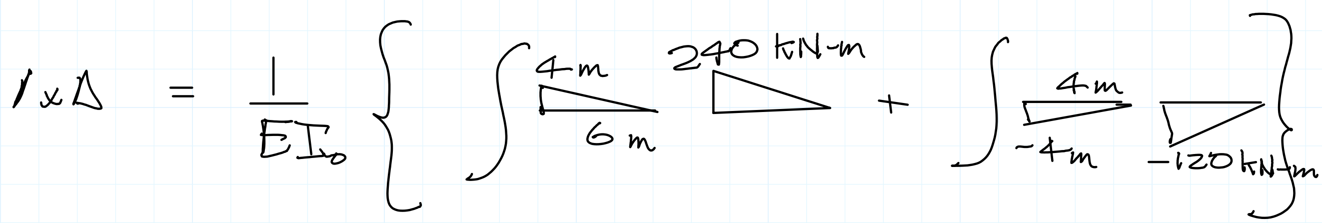
$I_o = 500 \times 10^6\ mm^4$

$K = 1333$ kN/m

Compute vertical displacement of pt. c.

Contribution of flexure:



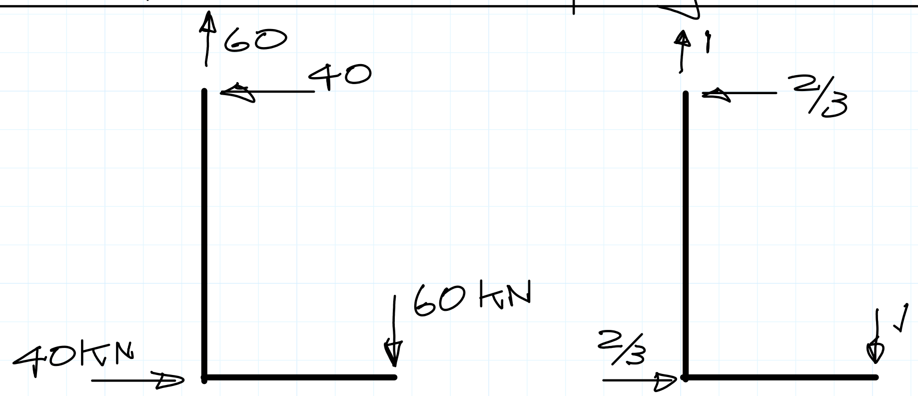


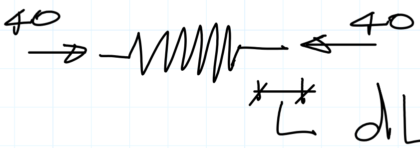
$$= \frac{1}{EI_o}\left\{\frac{6}{3} \times 4 \times 240 \underset{kN\text{-}m^3}{} + \frac{4}{3} \times -4 \times -120 \underset{kN\text{-}m^3}{}\right\}$$

$$= \frac{2560\ kN\text{-}m^3}{EI_o}$$

$$= \frac{2560 \times 10^{12}\ N\text{-}mm^3}{200 \times 10^3 \frac{N}{mm^2} \times 500 \times 10^6\ mm^4} = 25.6\ mm$$

Contribution due to spring deformation





$$dL = \frac{F}{k} = \frac{40\ \text{kN}}{1333\ \text{kN/m}}$$

$$= 30 \times 10^{-3}\ \text{m} = 30\ \text{mm}$$

$$1(\Delta) = \sum u\, dL$$

$$= \frac{2}{3} \times 30\ \text{mm}$$

$$= 20\ \text{mm}$$

$\therefore$ Total Defln $= 25.6\ \text{mm} + 20\ \text{mm}$

$$= 45.6\ \text{mm}$$ ←